

OPENING PLENARY

STEPS TOWARDS A NEW EPISTEMOLOGY IN MATHEMATICS EDUCATION

Yves Chevallard, IUFM d'Aix-Marseille, France

Abstract: *This talk will centre on a number of inadequacies that beset mathematics education in present-day societies. In the author's view, the basic obstacles lie in – or can be expressed in terms of – the epistemological “regime” of the knowledge imparted by most scholastic institutions, as well as in its main social, cultural, and political correlates. It is therefore an essential responsibility – though not a monopoly! – of researchers in didactics to contribute to the advent of a new school epistemology, more in tune with the needs of our time – a crucial pursuit on which this presentation simply aims to shed some light.*

Madam Chairman, Ladies and Gentlemen, dear colleagues, allow me to thank the programme committee for its kind invitation to deliver the opening lecture of the 4th congress of the European Society for Research in Mathematics Education.

Let me first say a few words about my usual occupation in Marseilles, France, where I live and work. For several decades now, I have been involved in, at first, in-service teacher training and, later on – over the last twelve years or so –, pre-service teacher training. Mathematics teacher training is one of my two main commitments to mathematics education. The other one is doing *research* in mathematics education; or, more exactly, in the *didactics* of mathematics – a notion I shall comment upon in what follows. It is essentially from these two vantage points that I'll try to consider the situation – which is, in my view, a perplexing and difficult situation – of mathematics education today.

To do so, I'll try to determine what *didactics* is; or, more prudently, what it can be construed to be. According to dictionary definitions, the *noun* “didactics” refers to the science, or art, or profession of teaching – dictionaries refrain from choosing. It derives from the Greek *didaktikos*, which means (or meant) “skilful at teaching”. And it is akin (through Latin) to such words as *docile*, *doctor*, and *disciple*. The idea behind *didactics* is that someone attempts to do something so that someone – generally, someone *else* – learns something. The *adjective* “didactic” refers to a cultural posture existing from time immemorial. It is a posture so vividly identified in our European cultures that “didactic” has come to caricature what it normally simply

intends to depict – as when it is applied to someone striving to instruct someone else “even, as a dictionary puts it, when it is not welcome or not needed”.

What I shall henceforth call a *didactic fact* is any fact that can in some way be regarded as the effect of a socially situated wish to cause someone to learn something. Let me add – this is a more difficult point, on which I shall not dwell any longer – that a didactic fact is considered to be so only to the extent that it is effective in influencing the learning process. However, the meaning I shall assign to the noun *didactics* will be a little more liberal, with a view to encompassing an even wider range of phenomena. Didactics should, in my view, be defined as *the science of the diffusion of knowledge* in any social group, such as a class of pupils, society at large, etc. This “definition” requires some comments. In the first place, let me emphasise that its referring to a *science* is no writing automatism. It points to the fact that research – in mathematics education, for example – *is not enough*. Science is both a *process* of gaining knowledge, and the *organised body of knowledge* gained by this process. (It happens that, in didactics, the knowledge gained and organised is about... the diffusion of knowledge!) Doing didactics is therefore not only just “doing research”, and, consequently, producing pieces of knowledge; it is also, inseparably, organising these pieces into a body of knowledge – *didactics* –, with an experimental (or clinical) basis *and* a theoretical superstructure endowed with a paradoxical capacity, that of strengthening its empirical foundation. The true social aim of research is to make new knowledge available to the world. This indeed is a lofty goal, but without it doing research would be almost entirely useless. In this respect, allow me to conjure up with gratitude and esteem the tall, elegant figure of that prominent mathematics educator, Hans-Georg Steiner, who passed away a few months ago, and who so aptly argued in support of a *theory* of mathematics education.

My second comment will be about the *nature* of the knowledge whose diffusion will be studied. The answer to this question can certainly be expressed in terms of “bodies” of knowledge: if we do so, didactics becomes the scientific study of how bodies of knowledge percolate through human groups. This is essentially the formulation I used, a quarter of a century ago, within the framework of the didactic transposition theory. In order to go further, however, one has to raise an almost puerile question, which is: the knowledge whose percolation is to be studied is the knowledge *of what*? In other words, what is the *object* of that knowledge? My answer will be formulated in terms of a key notion that I’ll have to describe to some extent: the notion of *praxeology*. Some dictionaries define praxeology as the study of human action and conduct. Up to a point, this is not foreign to the use I will make of that key word of the anthropological approach to didactics – provided we include in “praxeology” the study, not only of what people *do*, and *how* they do it, but also of what they *think*, and *how* they do so. In that sense, didactics includes praxeology, or at least some part of it, because the knowledge percolating through society is about human ways of doing and thinking: the didactics of mathematics, for example, is bound to accommodate a “praxeology of mathematics”, that is, a scientific

description and analysis of what we, human beings, do and think when we “do mathematics”. But what I shall call *a* praxeology is, in some way, the basic unit into which one can analyse human action at large. (The concept of *a* praxeology is therefore basic to praxeology as a science – in the dictionaries’ definition of the word.) What exactly is *a* praxeology? We can rely on etymology to guide us here – one can analyse any human doing into two main, interrelated components: *praxis*, i.e. the practical part, on the one hand, and *logos*, on the other hand. “*Logos*” is a Greek word which, from pre-Socratic times, has been used steadily to refer to human thinking and reasoning – particularly about the cosmos. Let me represent the “praxis” or practical part by *P*, and the “logos” or noetic or intellectual part by *L*, so that a praxeology can be represented by [*P/L*]. How are *P* and *L* interrelated within the praxeology [*P/L*], and how do they affect one another? The answer draws on one fundamental principle of ATD – the anthropological theory of the didactic –, according to which no human action can exist without being, at least partially, “explained”, made “intelligible”, “justified”, “accounted for”, in whatever style of “reasoning” such an explanation or justification may be cast. *Praxis* thus entails *logos* which in turn backs up *praxis*. For *praxis* needs support – just because, in the long run, no human doing goes unquestioned. Of course, a praxeology may be a *bad* one, with its “praxis” part being made of an inefficient technique – “technique” is here the official word for a “way of doing” –, and its “logos” component consisting almost entirely of sheer nonsense – at least from the praxeologist’s point of view!

Let me add here two or three more remarks. First, in the anthropological approach to which I have contributed for more than two decades, all human forms of activity are supposed to result from the bringing into play of praxeologies. When I blow my nose, for example, I draw upon some praxeology, which may vary according to the culture in which I was brought up. When I walk, I also put some praxeology to use, and this praxeology may well vary according to gender, milieu, and so on. Following the French anthropologist Marcel Mauss (1872-1950), I will say that a praxeology is a “social idiosyncrasy”, that is, an organised way of doing and thinking contrived within a given society – people don’t walk, let alone blow their nose, the same way around the world. My second remark builds on the first one: the concept of praxeology is a generalisation of the notion used previously, that of “body of knowledge”. Indeed, many or even most praxeologies of ordinary life are denied the status of “body of knowledge” – who would accept that blowing one’s nose or walking in a park means bringing some duly learnt “body of knowledge” into play? (Up to a point, I would!) In the following, I shall therefore stick to describing human action in terms of praxeologies, without inquiring whether people generally regard them as “true” bodies of knowledge or as simple know-how, or even as a “natural” endowment – most people think breathing is something natural, for example. Third remark: human praxeologies are open to change, adaptation, and improvement. If I have to write the number

$$A = \left(\frac{2}{1 + \sqrt{3}} \right)^4$$

in standard form (i.e. $a + b\sqrt{3}$, where a, b are rational numbers), I can know *that* $x = 1 + \sqrt{3}$ is a non-zero root of a quadratic equation, and I can know *how* to generate this equation, which is $(x - 1)^2 = 3$, or $x^2 - 2x = 2$. It then follows that

$$\frac{2}{x^2} = 1 - \frac{2}{x} = 1 - (x - 2) = 3 - x,$$

and therefore that

$$\frac{4}{x^4} = 9 - 6x + x^2 = 9 - 4x + 2 = 11 - 4(1 + \sqrt{3}) = 7 - 4\sqrt{3},$$

so that $A = \frac{16}{x^4} = 4(7 - 4\sqrt{3}) = 28 - 16\sqrt{3}$.

Let me indulge in one more example. Marta's car uses 35 litres of unleaded petrol to drive 320 kilometres; how many litres will she use to drive 640 kilometres? In this case, the answer is easy – Marta's car will use *2 times* 35 litres, that is to say 70 litres. But what about driving 950 kilometres? Most non-mathematical people, I suspect, would recoil at the idea of confronting such a problem! Now, the following, very simple technique will do the trick: to do a number x of times 320 kilometres, the car uses x times 35 litres of petrol; so, to do 950 kilometres, that is to say $\frac{950}{320}$ “*times*”

320 kilometres, the car will use $\frac{950}{320}$ times 35 litres, which, according to the calculator in my cell phone, equals 103.90625 litres, or approximately 104 litres of (unleaded) petrol!

The preceding technique is the key component of the “praxis” part of an arithmetical praxeology useful whenever proportionality is involved, and which, therefore, will make life more pleasant, closer to “good life”. Making life more pleasant is, since classical antiquity at least, the main pursuit of all peoples and cultures. (Everyone knows, I presume, the assertion in the preamble to the “*unanimous Declaration of the thirteen united States of America*”, adopted on the 4th of July, 1776, that all men are endowed “with certain unalienable Rights”, and that, “among these are Life, Liberty and the pursuit of Happiness”. The preamble to the “Treaty establishing a Constitution for Europe” less boldly puts forward that “The Union's aim is to promote peace, its values and the well-being of its peoples”...) Now it is worthy of note that, on the subject of proportionality, the ancient Greeks could never reach – if I may say so – the technique I presented above, because their *logos* could not accept the metaphor that makes a fraction into a full-fledged number and allows one to speak of a *fractional* number of *times* exactly as if it were a *whole* number of times. In this respect, the ancient Greeks never found the “North West passage” leading from whole “natural” numbers to “artificial” numbers.

All praxeologies, and among them those intended to make life better for all, are “artificial”, that is, products of human cultures. The age-old, ever recurring bias against things contrived “by art” seems to be a permanent cultural trait across the world and throughout mankind’s history. Human societies take fright at humanly creations, as if these anthropic additions to nature were offences against the orderly cosmos, and a misdeed for which societies should punish themselves by decrying and, eventually, rejecting the artefacts they bring to life as steadfastly as they repress them. Surprising though it may sound, this repression is, in my view, basic to what happens in our societies as concerns the diffusion of knowledge, that is, the diffusion of praxeologies, which are the very substance human action is made of. Praxeologies are artefacts, and, conversely, all artefacts are praxeologies or *ingredients* of praxeologies invented to give praxeologies flesh and bones. (A city, or a theatre, are ingredients of praxeologies which allow us, respectively, to live together “urbanely”, and to flock together for leisure or instruction...) Now the ambivalence towards artefacts engenders a kind of splitting of artefacts, and therefore of praxeologies, each of them being split into a “good” object and a “bad” one. This is a crucial point. The *bad* one is the praxeology as such, that is, an organised body of notions, ideas, statements, justifications and explanations – the *logos* part – and of ways of performing a certain *type of tasks* (solving quadratic equations, blowing one’s nose, composing a fugue, or achieving no matter what), which make up the “praxis” part. The *good* object is the praxeology once we have blanked out what it was here for, that is the praxeology deprived of its inbuilt uses. In this fashion, praxeologies are soon turned into *monuments*, that is, things notable or great, fine or distinguished, but which, paradoxically, are effective in helping us to forget what they stand for – what exactly was the thing “monumentalised”. Everyone has heard, I suppose, the urban legend about the student who, having to solve a quadratic equation whose discriminant was 7, said: “If 7 is less than zero, then the equation has non-real, complex conjugate roots; if 7 equals zero, the equation has one real, double root; and if 7 is greater than zero, the equation has two real, different roots.” This indeed expresses something genuine about the epistemological regime of mathematical praxeologies diffused at school. These praxeologies are hardly instruments devised to gain insight into types of mathematical situations and to operate efficiently in those situations. They are rather bodies of knowledge that the student has to “visit”, and, if possible, honour and praise. The urban legend “If $7 = 0$ ” is a gross distortion of the naked truth. Still, the prevailing mode of turning a taught praxeology into a school monument consists indeed in cutting it off, more or less surreptitiously, from the authentic mathematical situations whose treatment might reasonably call forth the praxeology in question. In this way, school propagates a relation to knowledge close to fetishism. Praxeologies are accordingly studied not for what they would allow us to do or to think, but for themselves. It’s knowledge for the sake of knowledge, and even, if I dare say so, know-how for know-how’s sake!

There is an easy way to make the current, “monumentalistic” school epistemology visible – by asking for the *reasons* why such and such praxeology or such and such

praxeological “ingredient” exists. Why do mathematicians seem so attracted to *triangles* for example? Why does geometry tell us about *angles*, *lines* and *rays*, or about *crossing lines* and *parallel lines*? Why does geometry make room for the notions of *acute angle*, *obtuse angle*, and *reflex angle*? If you are tempted to answer: “Mathematicians are interested in all these entities simply because *there do exist* crossing lines, rays, acute angles, reflex angles, etc., that is, just because these ‘things’ are *out there*, in the natural world, waiting for us to study them”, then you have been infected with the evil “monumentalistic” doctrine that pervades contemporary school epistemology. If indeed you accept such a poor, unspecific reply, it is more than likely that you have secretly espoused a naturalistic view of the human world – including the mathematical world –, forgetting that almost everything out there, as well as everything in our minds, is socially contrived. A straight line is a concept, not a reality outside us. It is something created in order to make sense of the outside world and to allow us to think and act more in tune with that reality. When dissonance grows too much, we invent – well, *some* people invent – a renewed *logos* and a changed *praxis*. Fractal geometry, for example, speaks differently about the same “given” world, for it goes far beyond the concept where Euclidean geometry stopped – the fictitious straight line. For every praxeology or praxeological ingredient chosen to be taught, the new epistemology should in the first place make clear that this ingredient is in no way a given, or a pure echo of something out there, but a purposeful human construct. And it should consequently bring to the fore what its *raisons d’être* are, that is, what its reasons are to be here, in front of us, waiting to be studied, mastered, and rightly utilised for the purpose it was created to serve. These are two necessary conditions for the diffusion of praxeologies to be meaningful. Why do we simplify fractions? What are the reasons for the seemingly irresistible urge to reduce them to lowest terms? Likewise, what are the reasons that, in some situations, make us speak in *percents*, which is almost the opposite of expressing a fraction in lowest terms? And, to crown it all, why do we spend so much time visiting that impressive and apparently inescapable monument called “Converting Fractions, Decimals, and Percents”? All these questions will have to be duly answered. However, there is more to it than that.

To take a global view of the problem school is faced with, we must consider a four-character play. The first character is society, the second is good life (or happiness, or well-being), the third is the bulk of praxeologies already existing or still waiting to be created, and the fourth is school. Society endeavours to achieve conditions of well-being for its members, and especially for its younger generations, through the creation and subsequent diffusion of praxeologies, thereby trying to put the right knowledge into the right place. There are mainly two ways to do that. The first has been amply criticised: it proceeds by diffusing praxeologies deprived of their *raisons d’être*, as if praxeologies were meaningful by themselves. The second way will sound familiar to whoever has once really called upon some determined piece of knowledge or know-how to achieve something, be it in scientific research or in ordinary life. Praxeologies travel through society because they are necessitated to solve problems,

or, as I shall put it, to *answer questions*. The basic situation in this respect can be summed up like this: a *question Q* is raised, and an *answer A* is searched for. The question may be for example “How can we live together peacefully?” or “How can we work with large numbers?”, that is, numbers that the calculator in my cell phone refuses to take care of. These questions are “practical” questions, because answering them amounts to providing a “technique”, in the first case allowing people to live peacefully together, and, in the second case, allowing people to work effectively with large numbers. However, an answer cannot be reduced to just a *praxis*. We know *praxis (P)* eventually calls for some form of *logos (L)*, so that any answer is to be thought of as a part of a whole praxeology. Roughly speaking, an answer *is* a praxeology of a sort. And the migration of praxeologies through society can be explained in terms of questions and answers. In a given institution *I*, a question *Q* emerges; people in the institution seek an answer *A* to *Q*, that is, an adequate praxeology $A = [P/L]$. Generally, some supposed variant of the wanted praxeology exists somewhere within society. The people in the institution therefore have to locate that praxeology, and then make a copy of it. “Copying” is not the right word here. What happens is a *reconstruction* process that I called – years ago – a process of *transposition*. The original praxeology, let me call it $[\Pi/\Lambda]$, is transposed into a new praxeology, $[P/L] = [\Pi^*/\Lambda^*]$, supposed to be better at surviving the constraints imposed on both its “praxis” part Π^* and its “logos” part Λ^* by its new habitat, *I*. The *raison d’être* of praxeology $[P/L]$, the reason why this praxeology is now present in *I*, becomes clear: it has been brought into this institution because it was expected to solve a problem, to answer a question. It was wanted for just that reason – not for itself, however sophisticated it is.

The two questions I took as examples show that an answer *A* to a question *Q* does not always exist, and that, when it does, uniqueness is not sure. There certainly exist many “ways of life” to ensure that people will *not* live in peace, but we know of no way of life that would be a complete and perfect answer to the first question raised. Likewise, we do not doubt, I suppose, that there are several ways, not all easy, of working with large numbers. It is now time for me to introduce the third character, school. School is a manifold concept. As you probably know, the words “school” in English, “escola” in Catalan, “escuela” in Spanish, “école” in French, etc., all go back via classical Latin *schola* to Greek *skholè*. Originally, *skholè* meant “leisure” and gradually developed through “leisure used for intellectual argument” to “studious leisure, study”. The idea I would like to put forward is that, following in the wake of the ancient Greeks, European societies and their many institutions equipped themselves with different forms of *skholè*, that is, with institutions designed to allow for that specific need of human groups, finding answers to questions that beset them. A *skholè*, if I may say so, is therefore organised around the study of a number of questions *Q* to which the *skholè*’s students seek to give answers *A*. Two aspects have to be emphasised here. First, for that “scholastic” process of study not to be imposed on the students, it is necessary for the questions *Q* to be regarded – by the students,

by their teachers, and, so to speak, by the *skhole*'s "board of trustees" – as *crucial* to a better understanding and mastery of their lived world. Second, in studying questions Q , students will have to investigate many other, *derived* questions Q' , dynamically raised by the study of Q . Addressing these derived questions will lead to the transposition of many praxeologies A' , that will answer many unintended questions. In the long run, this basic phenomenon will turn the students (and their teachers) into "scholars" of a sort. For example, in studying some naïve question of ecology, Q , supposedly crucial to the well-being of their region, students may be led to learn a little bit of difference equations and a lot about photosynthesis, and something about many other subjects. However, a third point must be made clear. A major principle in trying to establish a new epistemology at school is that one should *not* go directly to the questions Q' , let alone the questions Q'' generated by the study of Q' – unless questions Q' or Q'' seem crucial to the *skhole*'s students and teachers. To go straight to a question Q' without being motivated to do so by the study of a previous, crucial question Q generally means that the study process is going adrift and will soon be replaced by the mere inspection of a succession of official monuments of knowledge, that is, the monumentalised praxeologies normally called forth by the study of question Q' .

This description of what should take place at school according to the new epistemology I refer to leaves the question of how to implement such a new epistemological and didactic "regime" open. One vital point here is that questions Q should be taken *seriously*, not as mere opportunities, soon forgotten, to bring up the study of some predetermined mathematical monuments, the way an illusionist conjures a rabbit out of a top hat. For several reasons, among them the wish not to surrender to the opportunistic spirit which pervades the old school epistemology, I have propounded a type of didactic structure that I tentatively propose to call, in English, a *study & research programme*. The French name is actually "*parcours d'étude et de recherche*", "study & research course", where course is to be taken in the sense, say, of a golf-course. A S&R programme is to be thought of as a part of the curriculum, together with several other such programmes. It is essentially determined essentially by the will to bring an answer, A , to some *generating question*, Q , but it is also determined by constraints imposed upon the study & research to be done by the existing curriculum – which, for example, will not allow the class to draw upon such and such advanced mathematical praxeology. Up to a point, then, it can be said that a S&R programme is underdetermined, or that it is a context-bound scheme. However, such a situation is not at all peculiar to the school management of knowledge: in any research lab, all over the world, what is going on does not depend only on the question studied, but also on all sorts of resources – including intellectual resources – that the lab can obtain. The relative lack of determination inherent in the notion of a S&R programme turns the study & research process which question Q generates (and not only starts up) into what study & research should be – an intellectual, human, and institutional *adventure*, which may develop along different routes, within the territory bounded by the curriculum.

Research on the didactic technology (and theory) of S&R programmes (or courses) is currently one of my main concerns as a researcher in didactics. This pursuit leads almost immediately to another key problem in the anthropological approach to didactics – a problem I have labelled the *dialectic of media and milieus*. Before I explain this terminology, let me say a few words about the joint use of “study” and “research” when it comes to labelling S&R programmes. “Study” is used here as a comprehensive term, meant to include research; but study at school often precludes research, if one understands this word in the sense I shall now try to make clear to you. The editor of a book entitled *The pleasure of finding things out*, gathering the “best short works” of Richard Feynman (1965 Nobel Prize in physics), writes in his introduction: “Another of the most exciting events, if not in my life, at least in my publishing career, was finding the long-buried, never-before-published transcript of three lectures Feynman gave at the University of Washington in the early 1960s, which became the book *The Meaning of It All*; but that was more the pleasure of finding things than the pleasure of finding things *out*.” Now it seems that most school study consists in *findings things* – as is the case with “documentary research” for example –, that is, finding works that can, equally well, either provide the needed praxeologies (in the framework of a S&R programme) or be turned into monuments that one will visit without even trying to find *out* what they are here for. The mention of “research” in “S&R programme” is intended to convey the idea that such a programme is designed to allow students to do research, that is, to “find things out”. This is where I can take up the “dialectic of media and milieus”. Students – and all of us indeed – are surrounded by *media*, a word I use here in a generalised sense, calling “medium” *any* social system pretending to inform some segment of the population or some group of people about the natural or social world. In such a comprehensive view, a course of lectures, for example, is a medium, and so is a textbook, and the same can be said of urban legends passed on by word of mouth... The problem that arises here can provocatively be formulated thus: how can a student ascertain that his/her teacher’s claims are not a sheer succession of fallacies and pieces of misinformation? This is where the notion of “milieu” comes in. “Milieu” should be understood here, not in its sociological sense, but in the sense given in Guy Brousseau’s theory of didactic situations to the concept of “a-didactic milieu”. To make a long story short, let me say that I call “milieu” any system that, as far as the question that you put to it is concerned, is devoid of intentions and therefore behaves like a fragment of nature – a system that intends neither to please or to displease you nor to defeat you of your hopes. In mathematics, of course, proofs are the chief traditional milieu in that sense – a deductive system does not try to comply with the mathematician’s wish... The dialectic of media and milieus is in my view the central problem of our time, at school *and elsewhere*, in building a *democratic* epistemological regime. Such a new social relation to knowledge should at long last eliminate the ubiquitous remnants of the social epistemology that flourished in what is known, in the history of my country, as the “*Ancien Régime*” (“Old regime”), that is, the regime that prevailed before the French Revolution, in which there existed, in

the classroom and elsewhere, almost no other milieu, at least for the general public, than the “master”, regarded as an authoritative source that, in retrospect, appears to be a poorly a-didactic milieu!

For a S&R programme to be effective as a means to get the younger generations to tackle a number of questions of interest to them and to society, a generating question Q must not only be “crucial” (and therefore legitimate). It must also have sufficient “generative power” to engender *many* questions open to study and research. In that respect, choosing generating questions is a crucial step. Who will take that step? It should be clear that this is where politics comes in. Choosing the generating questions and the main routes of study and research is at the same time a curricular and political matter, so that it is not up to researchers and educators alone to decide, even if they can voice their own views as experts and as citizens. However, I would like to conclude by adding a very short note on a too often misunderstood political concept, without which, in my view, it remains impossible to fully achieve democracy, namely the so-called “French” notion of *laïcité*, a concept that cannot be reduced to secularism, but that must on the contrary be *extended* much beyond religious considerations. According to that political principle, no vision of “good life” may be *imposed* – however surreptitiously – *on anyone*. A vision of “good life” usually includes views about religious matters; but it also includes views about “earthly” matters – literature, music, mathematics, and so on. The principle of *laïcité*, when applied rigorously, implies that, even in mathematics, when questions Q are raised and answers A are obtained, while it is legitimate to require of the students that they “know” the corresponding praxeologies and their *raison d’être* (and to be able to bring them into play relevantly when asked to do so), it would be *utterly illegitimate* to urge them to regard questions Q and, even more, answers A as, respectively, the right questions and the right answers to make life better – including mathematical life! Mathematics educators, in particular, are not asked to make students *love* mathematics, nor to hate it of course, but to *know* mathematics, which is quite a different and demanding task! Love, hate and indifference reside in each and every one of us. To ignore that principle would be in my view *utterly undemocratic* – not exactly what school is supposed to achieve.