# IMPLICIT MATHEMATICS:
# ITS IMPACT ON SOCIETAL NEEDS AND DEMANDS

by Yves Chevallard
IREM d'Aix-Marseille
70, Route Leon-Lachamp
13288 Marseille Cedex 9 (France)

My presentation will center on a few, bold contentions, that I shall try to substantiate concisely. To begin with, let me say that, in essence, I shall dispute the validity of a wide-spread belief, forcefully expressed in the following quotation borrowed from ICME V's report on mathematical modelling [1]:

> The ultimate reason for teaching mathematics to students, at all educational levels, is that mathematics is useful in practical and scientific enterprises in society.

The following considerations aim to reveal and, to a certain extent, to explain the essential ambiguity in this and similar declarations.

## 1. On the alleged utility of mathematics

My main contention in this regard can be expressed tersely.

1.1. No modern society can live *without* mathematics.

1.2. In contradistinction to societies as organized bodies, all but a few of their members can and do live a gentle, contented life *without any mathematics whatsoever*.

Certainly both theses require careful explanation. They seem to refer to the "degree of presence" of mathematics in society. But to gain insight, one must resort to another, distinct notion, that of the *mode of presence* of mathematics: mathematics may be present either in explicit form or in implicit form.

## 2. Explicit (uses of) mathematics

*Explicit*, or *live* or *visible* mathematics, or more precisely its *explicit mode of presence*, is what people have in mind when they praise mathematics for being a necessity of life today.

2.1. Explicit mathematics is the mathematics that is visibly handled, used, manipulated in science (including mathematics), technology, engineering, business, administration.

[1] Proceedings of ICME 5, Theme Group 6: Applications and Modelling, p. 199.

2.2. Explicit uses of mathematics are essential to our present-day societies, but: 1. they are generally *concealed from public view*; 2. in going about their business, *most* people *never* meet with explicit uses of mathematics – save for some arithmetic [2].

2.3. Accordingly, not only does the word "mathematics" mean, for most people, explicit mathematics: it is usually reduced to apply to the only explicit mathematics that become normally visible to the layman, i.e. *school mathematics* [3].

2.4. Some mathematics educators claim – without much *direct* evidence – that "mathematics" permeates every aspect of life. As a further consequence, in so far as they want school mathematics to reflect faithfully the mathematics of the "outer world" as they see it, they become inveigled into drawing an image of reality stuffed with explicit mathematics, in order that reality may conform to the fiction they have created. Hence the plight of those who devote their energies to proving that mathematics, especially in the form of mathematical modelling, is "at work" in every nook and cranny of society – a Sisyphean labour and, up to a point, a wild-goose chase.

## 3. The social "implicitation" of mathematics into objects

If it is true that mathematics pervades present-day, Western-type societies, it is nevertheless true in a very different sense. The way in which mathematics penetrates our daily life is unremarkable, even banal. Mathematics in effect finds its shortest way to every one of us through "objects" of all kinds, in the form of *implicit mathematics*.

3.1. Implicit mathematics is formerly explicit mathematics that has become "embodied", "crystallized" or "frozen" in objects of all kinds – mathematical and non-mathematical, material and non-material –, for the production of which it has been used and "consumed".

3.2. The amount of implicit mathematics present in an objet, i.e. the amount of mathematics crystallized in it, can be roughly defined as the sum total of the explicit mathematics used in producing that object, and a fraction of the mathematics (previously) crystallized in the (material and non-material) "objects" consumed in the production of that same object [4].

3.3. Accordingly, beyond any object, however commonplace, one can invoke, in an infinite regress, all those fragments of mathematical knowledge and know-how which have been passed on from object to object, implicitly and most often invisibly, in the social production of, firstly, the object itself, secondly, the objects consumed in producing that object, thirdly, the objects consumed in producing the objects consumed to produce that object, and so forth [5].

3.4. The amount of crystallized mathematics present in a given object is exactly what I call the mathematical *grade* (or *content*, or *tenor*) of the object.

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[2] Gamblers use some combinatorics: but is not gambling a trade in its own right?
[3] In the mathematicians' sphere, mathematics is used in order to produce more mathematics; in the engineering sphere, mathematics is used to produce more knowledge and know-how of a different kind (in electronics for instance). In the case of school mathematics, mathematics is neither used nor produced: it is taught and learnt.
[4] The reader who, on reading this statement, is reminded even dimly of Marx's labour theory of value is sure not to have missed the point!
[5] This, as one may note, sounds like a typically recursive definition. The marxist flavour is again very insistent: see for instance Michio Morishima's *Marx's Labour Theorv of Value* (Cambridge University Press, 1973).

## 4. Mathematics is here around us

The implicit mode of presence of mathematics usually goes unnoticed; and such is the state and status of “mathematics” in the life of most people.

4.1. The amount of *explicit* mathematics used in producing an object *is almost always negligible*.

4.2. The amount of mathematics recursively crystallized in most objects – including the “necessities of life” – *is almost always considerable*: however negligible may be the amount of explicit mathematics used at any point in their social production process, it can result in the end in a considerable amount of mathematics crystallized in the “finished product” [6].

4.3. The degree of presence of a given means of production is in no way an intrinsic characteristic of a given type of object (as defined by its function and structure), but is actually *socially determined*, i.e. determined by the normal process of production prevalent at the time.

4.4. Although for example men of the Neolithic Age made stone implements – such as hammers and axes – which contained no mathematics at all [7], exactly the same objects with regard to both structure and function are now “stuffed” with crystallized mathematics. Such is the very reason why mathematics can be said to be everywhere around us, though in the unassuming form of dead, frozen mathematics, hidden in the multitudinous objects of everyday life.

## 5. The growing, invisible, silent presence of mathematics

While the live mathematics incorporated in a few social practices is often called on to speak, crystallized mathematics tells no tales: it is nonetheless an essential component of almost all things and situations that make up contemporary life.

5.1. The previous, much too concise account of the situation of mathematics in society is by no means peculiar to mathematics: considering the overall *production of society*, one can equally sensibly apply it to any material or non-material “means of production” of society, be it medicine, electricity, or steel, for instance.

5.2. To a certain extent, however, one can assess the growing importance and “utility” of mathematics in modern societies by a simple mental experiment: just as we could switch off electricity to satisfy ourselves that electricity is indeed a basic ingredient of developed societies, without which almost nothing would continue to exist, so also we can imagine that the “switching off” of mathematics would cause almost every socially produced thing to fail to exist – a fact coextensive with modern societies.

[6] That tiny fragments of mathematics can make up a significant whole is a fact that mathematically minded people should feel at ease about.

[7] With the exception of some religious monuments, in which traces of mathematics have been found, notably the supposed use of Pythagorean triples: see Bartel Leenert van der Waerden, *Geometry and Algebra in Ancient Civilizations* (Springer-Verfag, Berlin Heidelberg, 1983).

5.3. The degree of presence of implicit mathematics in society, i.e. the average mathematical grade or tenor of goods and services made available to the man in the street, has varied over time: a pervasive, centuries-old trend, linked to the development and formidable growth of science and technology, has for good or bad resulted *in a continuing rise in the mathematical grade* of objects. While for instance the content in steel of many necessities has dramatically fallen – think of motor-cars –, the empire of mathematics is steadily spreading and keeps encroaching on domains which until recently had remained foreign to its influence.

## 6. The dialectic between implicit and explicit mathematics

Social uses of mathematics lead to a paradoxical situation, which is more often ignored than analysed: one can say that, while society as a machinery is more and more mathematised, our daily life is more and more demathematised.

6.1. The greatest achievement of mathematics, one which is immediately geared to its intrinsic progress, can paradoxically be seen in the never-ending, two-fold process of (explicit) *demathematising* of social *practices* and (implicit) *mathematising* of socially produced *objects* and techniques.

6.2. This applies equally properly to *mathematical* practices and *mathematical* objects: whereas, for example, multiplication was held in Ancient Egypt to be a scholarly technique requiring much skill and intelligence, it has over time become so simplified – so "demathematised" – that even young children can now perform it – a fact so familiar to us that we usually do not question its meaning and significance.

6.3. The process of mathematisation/demathematisation is in fact the very foundation on which the social production of mathematical objects rests: while the mathematical *grade* of mathematical tools steadily *increases*, their mathematical *value* – that is, the average, socially determined mathematical labour-time needed to produce them – steadily *decreases*.

6.4. The process of demathematisation relates, of course, to the amount of explicit mathematics, i.e. of mathematical knowledge and know-how, needed to produce or to *use* [8] mathematical objects. As regards implicit mathematics, more and more objects tend to have a higher mathematical grade, thus becoming more and more *mathematically powerful*.

6.5. Both the rise in mathematical grade and the decrease in mathematical value must be invoked to explain the *social success of mathematics*: high mathematical grades make for the powerfulness and efficiency of the objects made available to us, be they material or non-material; and lower and lower mathematical values account for their *wide social availability*.

## 7. The individual dispensability of mathematics

I shall now take up the question why so few people are directly concemed with (explicit, live) mathematics; and why the ordinary citizen has in fact to deal with so little mathematics in his/her ordinary experience of society.

7.1. One cannot overstress the fact that, in achieving simplification, mathematicians and "mathematics workers" have constantly resorted to one single method. This method of "simplifying" mathematical objects consists in incorporating (explicit) mathematics into

[8] Possibly for the production of new mathematical objects.

them, i.e. in turning live mathematics into dead, crystallized mathematics: any ignoramus can now do any calculation whatever with his pocket calculator, and one should no longer worry even about the niceties of the addition of fractions (a state of things that, for the time being, many mathematics educators still resist).

7.2. The explicit and implicit mathematics embodied into any theorem or method (and which account for their increased powerfulness) are the mathematics needed to establish that theorem or method. But little or no knowledge of *those mathematics* is required in order to use relevantly the (mathematical) tool thus provided: one may use, as a mathematical tool, the theorem of Pythagoras without having any idea of anyone of its various potential proofs. Obviously, it is a regular outcome of the activity of mathematicians, throughout the centuries, that formerly difficult questions become easy or easier ones, and that mathematical tools which were at first the privilege of experts sooner or later become available to novices.

7.3. More generally, the increase in mathematical grade and decrease in mathematical value, and the ensuing increase in mathematical powerfulness and social availibility of mathematical objects, are coexistent with yet another, socially elemental characteristic: while the average mathematical grade of goods and services increases, *the average mathematical expertise required to consume those goods and services steadily decreases*.

7.4. This continuing line of historical development in the production of society explains why the average citizen simply does not have to care much – or very much – about mathematics, while mathematics is (implicitly) all around us in everyday life: most of the ordinary social practices in which he or she happens to take part have been deeply demathematised, a continuing process which has even accelerated since the advent of the microcomputer – thanks to which so many objects with high mathematical grade (and even with high mathematical added value) are made accessible to the "multitude".

**8. The cultural fragility of mathematics**

Those characteristics which explain the – more or less invisible – social success of mathematics also make for their *cultural fragility*.

8.1. Because it is generally concealed from public view, mathematics is scarcely given credit for what we owe it throughout our daily life. The social *effectiveness* of mathematics is essentially co-terminous with its social – and therefore cultural – *invisibility*.

8.2. The social debate on mathematics thus tends to center on *explicit* mathematics, and obscures the true role and major mode of presence of mathematics in society.

8.3. Moreover, the only explicit mathematics that most people ever come close to – in contradistinction to, e.g., the case of electricity – is *school mathematics,* i.e. mathematics as a subject-matter to be taught and learnt. That all but a few people experience explicit mathematics only under these conditions is a fact worthy of note, and the source of many societal problems for which appropriate solutions are yet wanting.

8.4. Indeed, the *teaching of mathematics* to the many is the way Western-type societies have tried to make mathematics *culturally visible*. The historical establishing of mathematics teaching could in fact be expected both to provide society with the necessary, mathematically skilled labour, and to achieve, on behalf of mathematics, cultural recognition and legitimacy.

(Because of its social invisibility, mathematics could not manage to survive socially without this recognition.) The venture, one must admit, has not been a complete success.

### 9. The case of the teaching of mathematics

In trying to reconcile society with mathematics, the central question to be answered, from which so many consequences flow, is: why has mathematics been obstinately taught at the secondary level (as opposed to the primary and tertiary levels), which is undoubtedly the weak link in our educational systems?

9.1. Certainly the growing empire of science over the life of Western societies, from the seventeenth century onwards, and the ensuing need for ever more engineers both civil and military, do account for the fact that *something* had to be done. But many examples – e.g., that of medicine – show that proper training of the required elites could have been started at the university level. That another, distinct line of action was decided on suggests that the main problem attacked was actually of a very different kind.

9.2. Because our societies need mathematics, because they are, so to speak, driven by mathematics, a balance had to be reached, in the sphere of culture, between society and mathematics. Society had, in some way or another, to recognize mathematics as a basic, major ingredient and driving force of economic and social development. By inculcating some mathematics in its children, society thus *paid a tribute to its needs* – to the increasing (implicit) mathematisation of society –, and one can reasonably doubt whether it will ever be out of debt in this respect.

9.3. To some degree, the recognition granted to mathematics proved misleading. For reasons still to be elucidated, the teaching of mathematics came to be justified in terms of the so-called "utility" of mathematics, and this in turn was understood *in terms of the individual's interests* – whereas the overall, *communal interests of society* as a global village were really at stake.

9.4. This determined, if I dare say so, a *cultural pathology* which not only misinterprets social needs of paramount importance, but may also take its toll on the pupil. The real nature of the problem facing us gradually faded from sight; accordingly, the solution afforded by the teaching of mathematics partly lost its intended efficiency.

### 10. How it all came about

The most essential problem that lastingly confronts the teaching of mathematics (and, more generally, the teaching of any subject-matter whatsoever) is the question of *its very existence as a social practice*, that is, the "socio-ontological" question.

10.1. Historically, people had to fight hard for the birth of mathematics teaching, and still have to attend to maintenance problems both corrective and preventive. One of the main assignments in this respect is to convince society as a whole that it *needs* mathematics teaching; or rather, that the teaching of mathematics is both necessary and most desirable.

10.2. In seeking to convince society of this vital idea, the pressure group that I call the *noosphere*, i.e. the people who devote time and energy to thinking about the teaching of mathematics, its present state and its foreseeable future, will try to impose simple views and, to this end, will propagate what I call an *apologetical discourse*. It should be emphasized that the apologetics of the noosphere has a very narrow thematic which, moreover, seems quite independent of the subject-matter on behalf of which it is proclaimed. In Western and Western-type societies at least, its recurrent, central claim is that the subject-matter in question should be taught and learnt, because both society as a whole and its members as individuals need to master it *in order to succeed* – success being appreciated according to changing criteria.

10.3. The balance between society's and the individual's reported needs may be achieved in many different ways. But in most cases modern societies have come to be infused with the peculiar spirit of what I shall term *individualistic democracy*, in which nothing can be entirely good for a given society unless it is presumed to be good also for anyone of its members. In such a context, those communal needs will tend to be ignored, and effectively neglected, which cannot be made to appear at the same time as common, personal needs – *as needs of the individual as such*. Hence the argument, so often resorted to by noospheric apologetics, according to which mathematics is useful *to almost everyone in almost all situations*.

10.4. Such an astounding privilege has been lavishly bestowed upon almost every subject-matter ever considered for teaching. It is part and parcel of the standard apologetical discourse that noospherians generally rely on. As often as not however, this kind of description wanders from reality; but, as I have tried to show, it is never more unrealistic and, if I may say so, ivory-towerish, than in the case of mathematics.

## 11. What's wrong with the current apologetical discourse

The quotation given earlier is in fact typical of statements proffered by the noosphere. The so-called "ultimate reason" offered here as an argument for teaching mathematics – its supposed usefulness "in practical and scientific enterprises in society" – is typically a noospheric reason, a reason which mingles two permanent and closely-related distinctive features: the impregnation of society with mathematics as a means of production, and the average citizen's personal relationship to mathematics as a body of knowledge.

11.1. Pronouncements in the noosphere, in fact, usually testify to the existence of some generally adverse set of conditions, of some problem with which the teaching system is confronted (or is likely to be confronted in the near future), and which it is the noosphere's duty to come to grips with.

11.2. At the same time however, noospherians usually – and interestedly – miss the point in such polemical declarations. They often cheerfully dismiss reality as it is and indulge in the fallacies of false consciousness. In other words, in voicing such declarations in defence of the teaching system, they make partially irrelevant strategic moves, whose side effects are generally unexpected.

11.3. As a counter-example to the usual noospheric argument, one might consider the case of medicine: medicine pervades our daily, "practical" life as well as major "scientific enterprises" – as the existence of industrial and forensic medicine, and the fresh growth of space and nuclear medicine show. For all that, it is not true that medicine is taught "at all educational levels".

11.4. More generally, should the utility of a given subject-matter be taken as a "reason" for teaching it "at all educational levels", such a reason would remain, obviously, an altogether insufficient one: considered as a subjective motive, it seems unconvincing, too weak in itself to act as a compelling force (most modern societies do not teach medicine at the primary and secondary levels, although medicine is held to be of paramount importance in most human activities); regarded as an objective cause, sufficient in itself to explain the historical establishing of the teaching of mathematics, and keeping the case of medicine in mind, one may wonder why, in this particular instance, like causes do not produce like effects. The social utility of a subject-matter is neither the ultimate reason for, nor the efficient cause of, its being taught. This conclusion, in my view, applies to mathematics as to other bodies of knowledge.

**12. Starting all over again**

It is up to us, I believe, to reconsider both the problem and the solution. It is my opinion also that in some sense, the noosphere will have to start all over again. And it will have to start at the beginning.

12.1. In choosing to fall back on that besieged territory – mathematics at school –, in pretending that it can serve as an appropriate base of operations from which mathematics could recover cultural visibility and achieve societal legitimacy, in arguing in the face of facts for the dubious utility of mathematics, the noosphere lacks either lucidity or courage – perhaps both.

12.2. I shall maintain that the teaching of mathematics at the secondary level is nothing but a means – for which, of course, we have to pay a high price – to reconcile culturally society with mathematics regarded as an inescapable societal need. Our societies may come to accept the idea that, in studying mathematics (as well as national history, the official language of one's country, etc.), everyone of us pays his personal contribution to the community.

12.3. The teaching of mathematics might then take on a new turn. It could keep closer to the true social role of mathematics and be made to play a more relevant part, that of *representing to the rising generation the way in which explicit mathematics is consumed in the production of society* – including the production of that essential component of society, mathematics. Like many other subject-matters taught at school, mathematics education at the primary and secondary levels should be relevantly defined as a *cultural initiation* – one which might enable all members of society to be in tune with the society to which they belong, to understand its most essential workings, and, as the case may be, to take an active part in its scientific and technological development.

12.4. Such an initiation should result in an awareness of society as a complex whole made up of many deeply-interrelated components, most of them hardly visible and understandable from the outside. It should avoid some major pitfalls, adress elemental, not necessarily elementary, questions, and beware of unrealistic realism. If it could cast off the sanctified fallacies that I earlier criticised, the "mathematics-at-work" movement might in this respect show us the way.